The dragon tile and other solutions for enclosure problems

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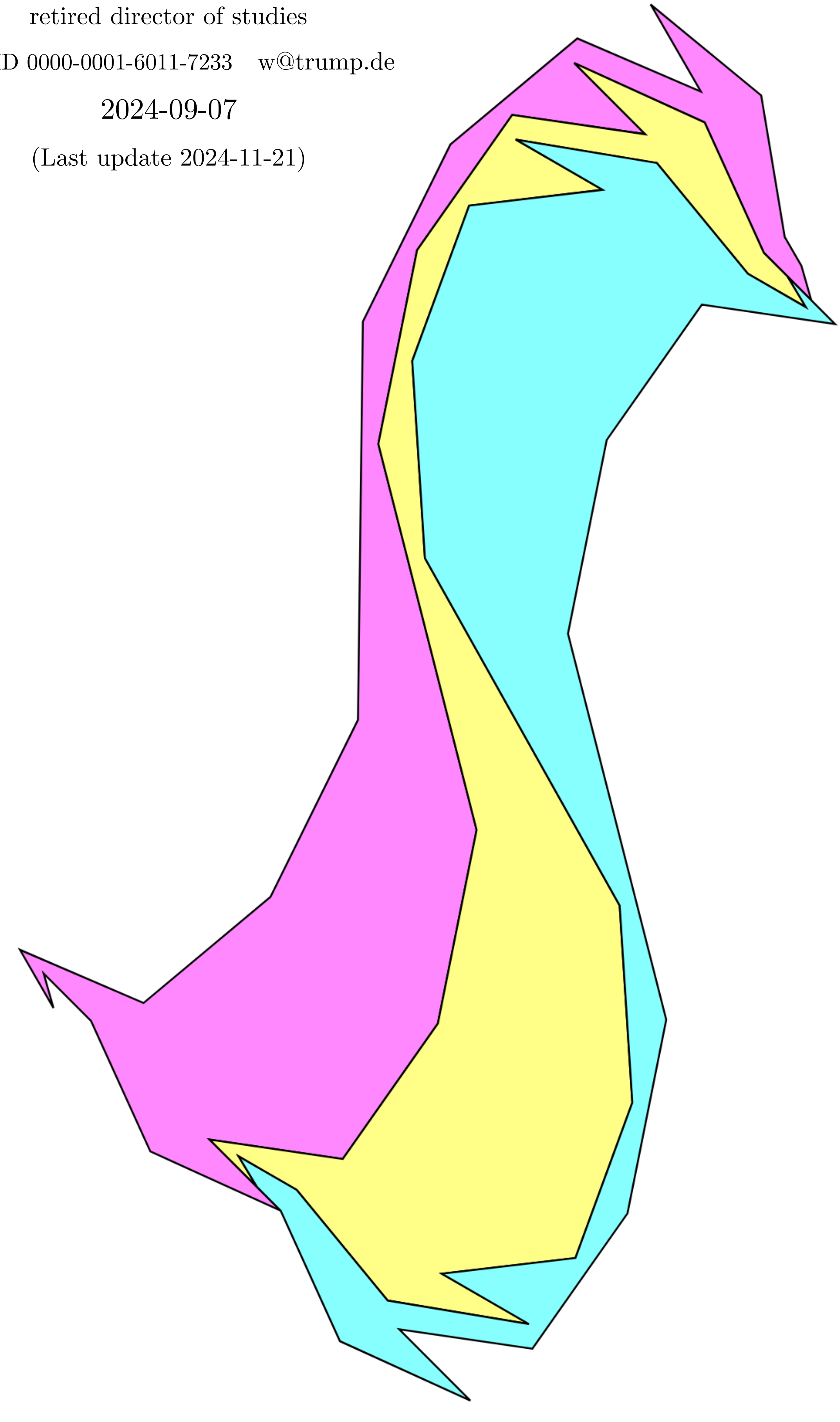

Abstract

New tiles are presented where two congruent tiles completely surround r copies of the same tile.
General solutions are shown that solve this problem for all integers r.
This is an early state publication. More details will be added over the next weeks.

Definitions

In this paper the definitions given in *The Tiling Book* of Colin Adams [4] are used.
For example a composite T of r tiles is (*completely*) *surrounded* by a finite set of other tiles if the whole tiling S is a topological disk and T has no point in common with the boundary of S.

1.1 The tiles of Heinz Voderberg

A Voderberg tile [1] can be *enveloped* (*incompletely surrounded*) by two copies of itself. But the tile is not completely surrounded by the two copies, because the two corners A and B of the inner yellow tile touch the boundary of the whole tiling.

Fig. 1 to 4 were constructed with GeoGebra [7].

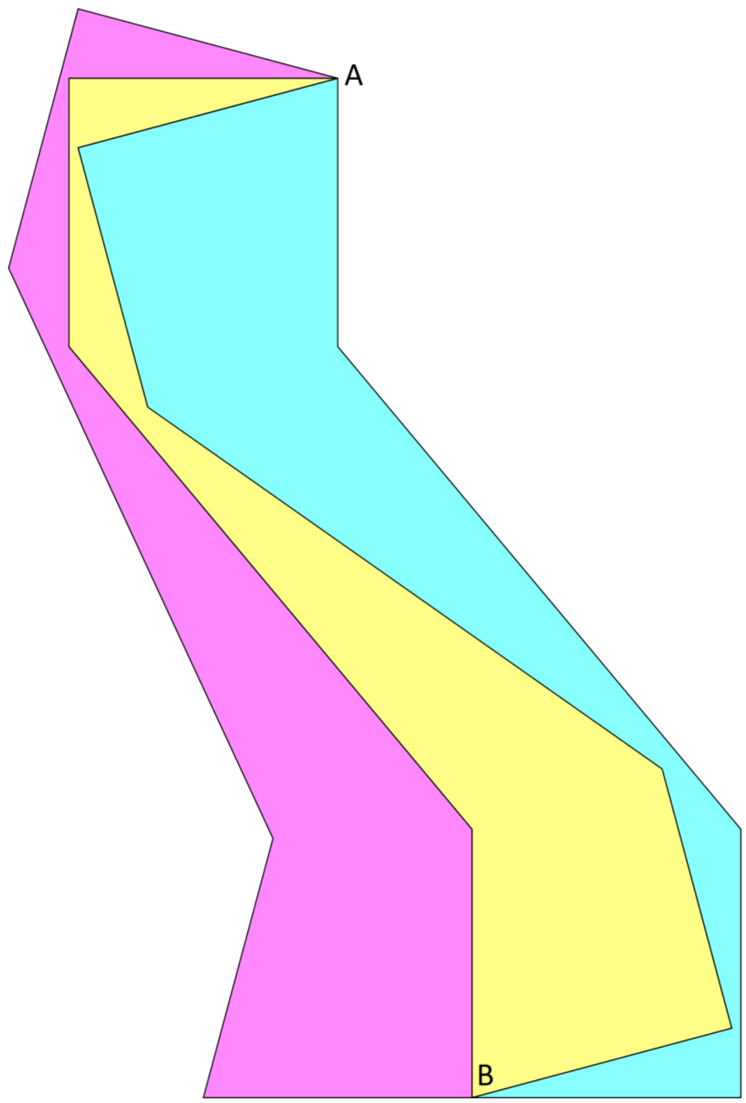

Fig. 1. A Voderberg tile (yellow) is enveloped by two copies of itself. (drawn by the author)

1.2 First solution for $r = 1$ by Casey Mann

In 2002 Casey Mann published the first tile with surround number 2. He modified a Voderberg tile by "strategic placement of some hooks and catches". [2]

The shown green tile has the same orientation as the yellow Voderberg tile in Fig. 1. Rotate the green tile by 15° (180°) in order to obtain the same Orientation as the violet (blue) Voderberg tile. See [5] for a modification with larger hooks and catches.

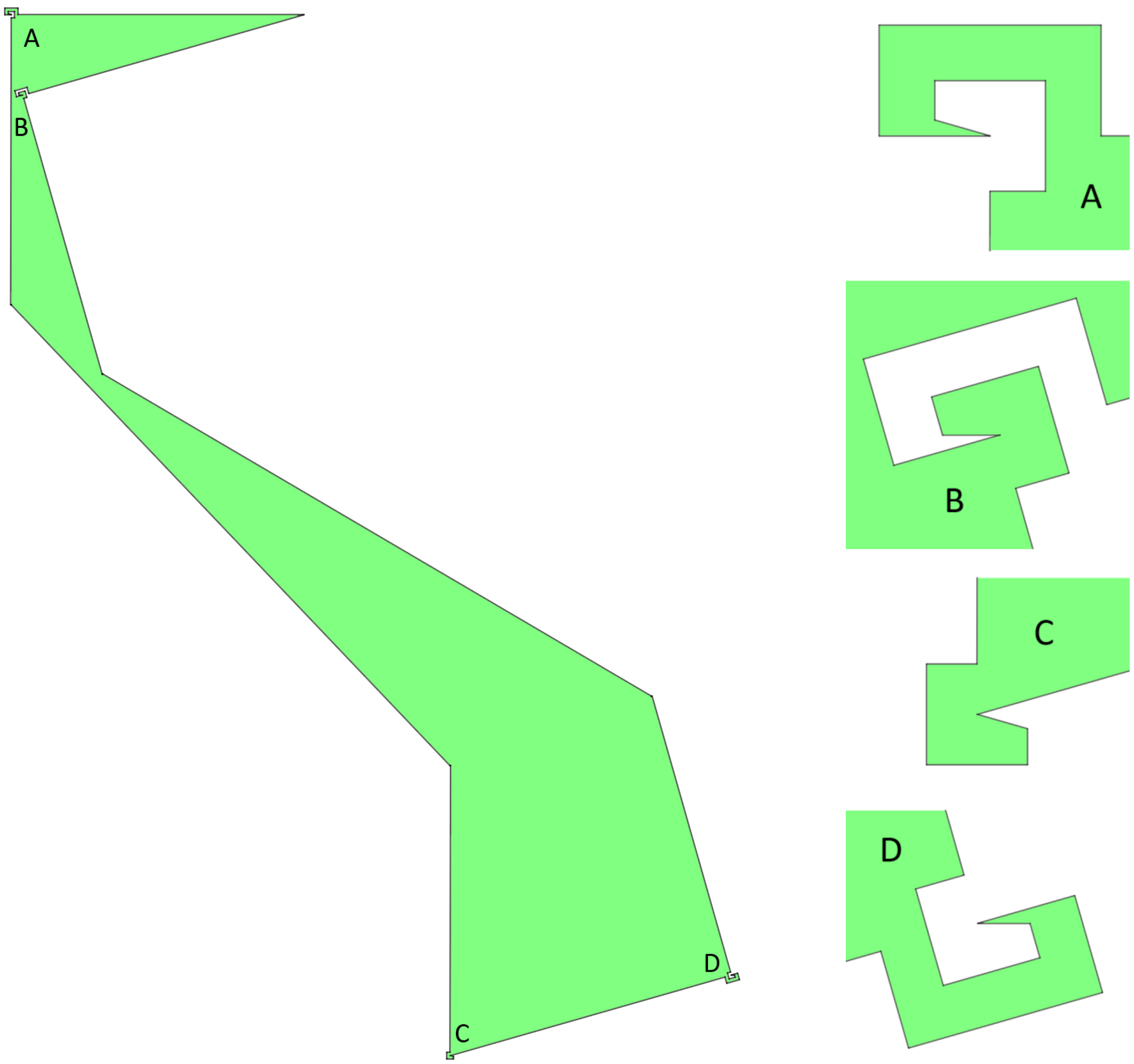

Fig. 2. Casey Mann's tile with microscopic details (drawn by the author)

1.3 The dragon tile ($r = 1$) additionally can tile the plane

Here a polygon with 29 edges (29-gon) is presented (Fig. 3), read in chapter 1.5 how it can be constructed. This tile can be completely surrounded by two copies of itself (Fig. 4) and it can tile the plane periodically and non-periodically (see chapter 1.4).

Many different variants of the tile are possible including 25-gons (Fig. 10). The yellow tile in Fig. 4 is achieved by rotating the violet tile by 15° counterclockwise. A rotation of the yellow tile by 180° leads to the blue tile.

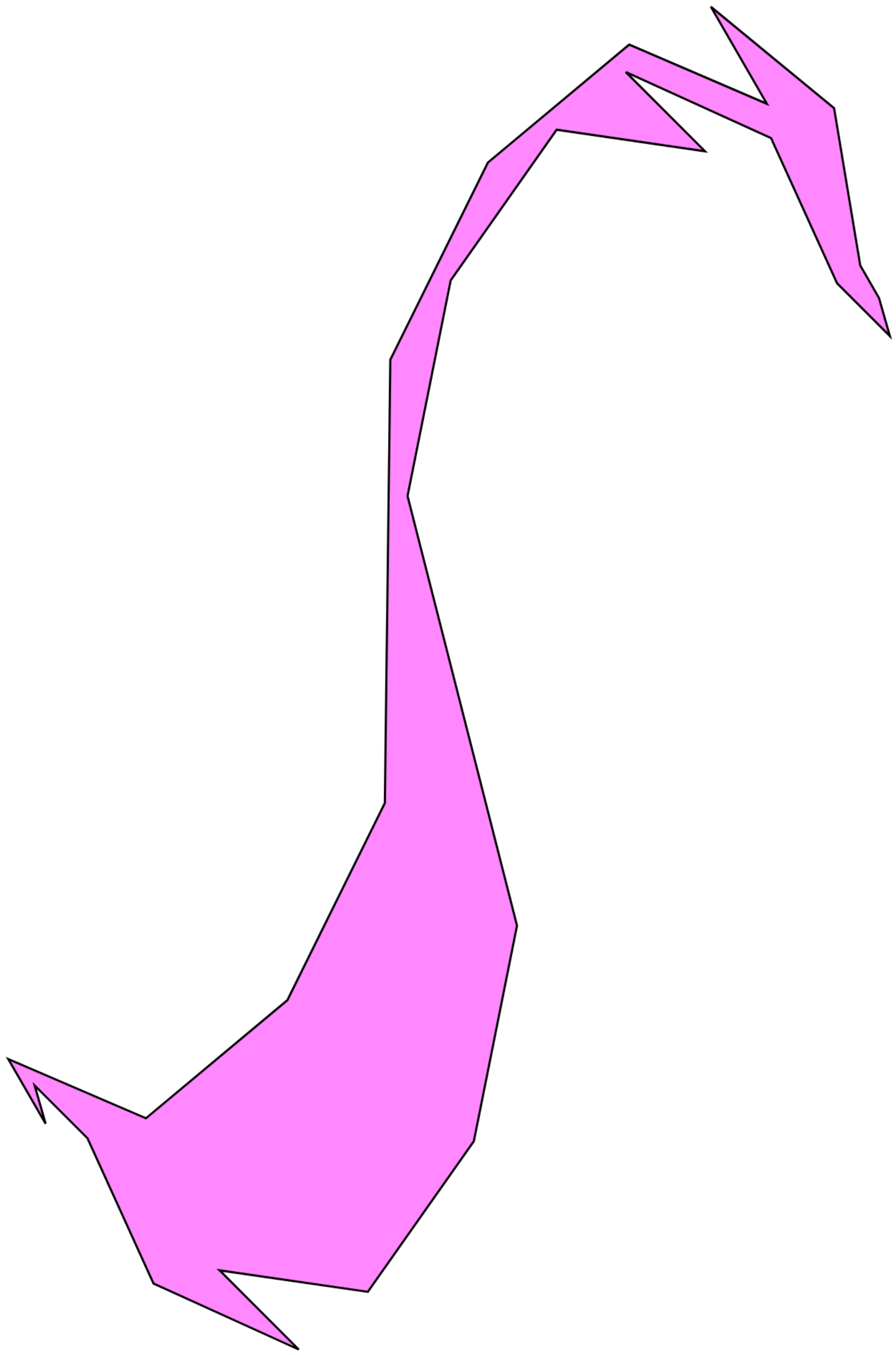

Fig. 3. The new tile with surround number 2, that additionally can tile the plane.

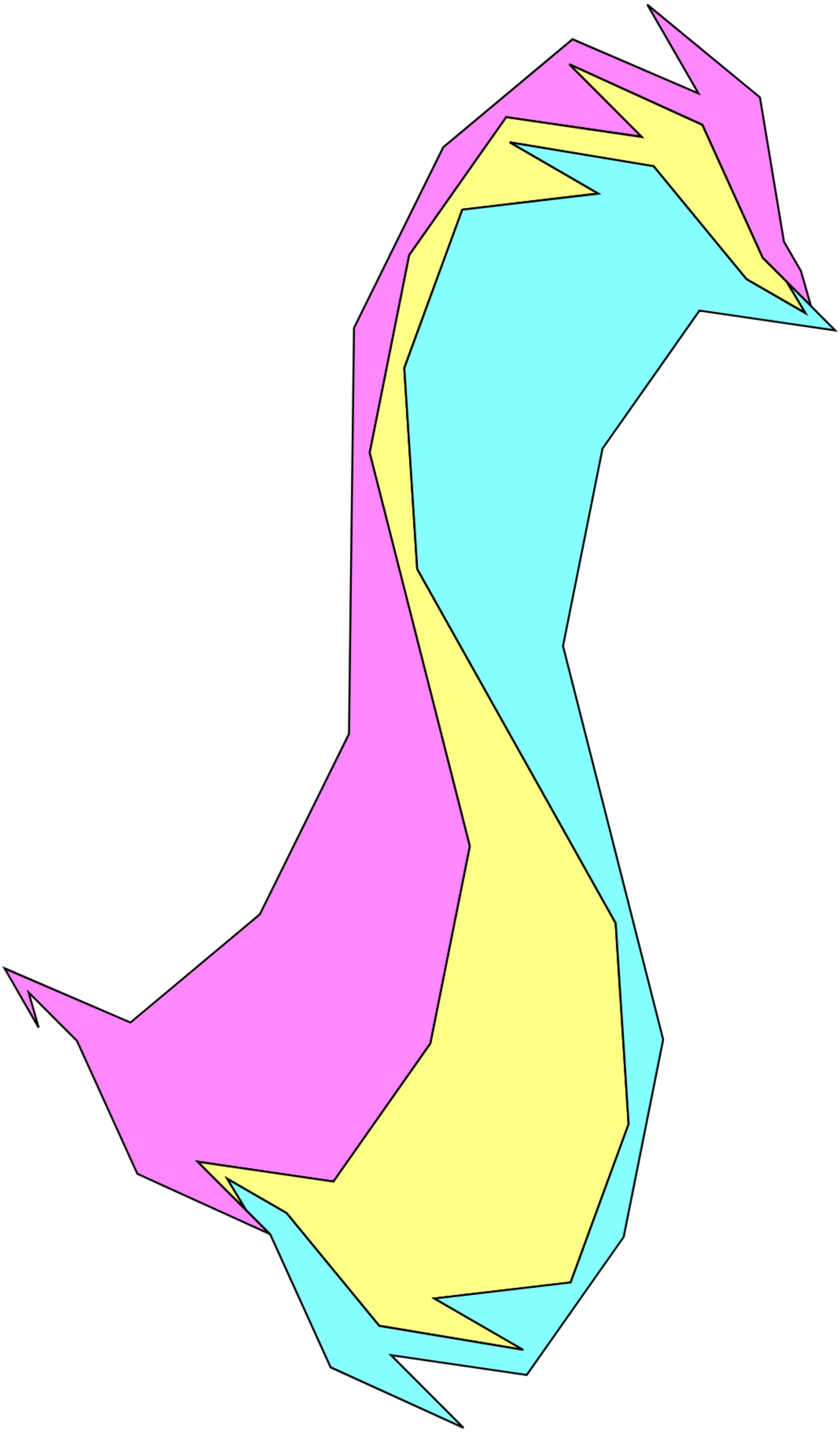

Fig. 4. The tile can be surrounded by two copies of itself. See [5] for SVG graphics.

1.4 Tiling the plane

The tile and a copy rotated by 180° form a polygon with twofold rotational symmetry (Fig. 5). An infinite stripe made of such tile couples has an upper and a lower boundary that is equal to the boundary of a sequence of isosceles triangles (Fig. 6). Thus, these stripes can be connected in order to fill the whole plane. This tiling is unique and periodic (Fig. 7). If we allow flips of the tile, a different stripe is possible and the two stripes *a* and *b* can be combined arbitrarily (Fig. 8). Thus, an infinite number of periodic and non-periodic tilings of the plane are possible. For example: Say *c* is any finite combination of stripes *a* and *b*. Then ...*cccc*... is a periodic tiling and ...*aaacaaa*... with only one *c* is non-periodic.

Fig. 5 to 8 were created with the app of Arnaud Chéritat [6].

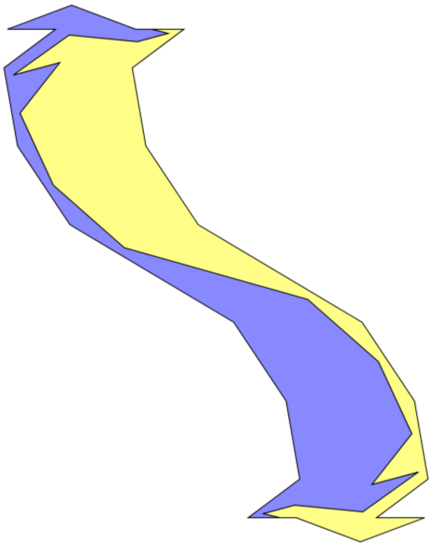

Fig. 5. Symmetric tile couple

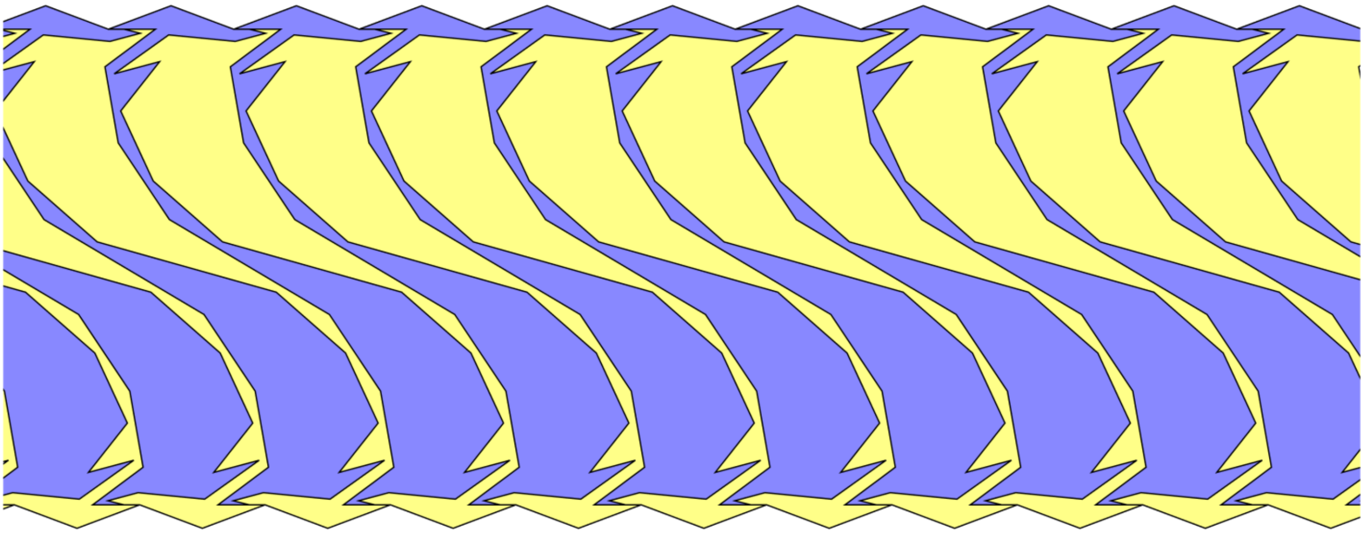

Fig. 6. An infinite stripe of tile couples

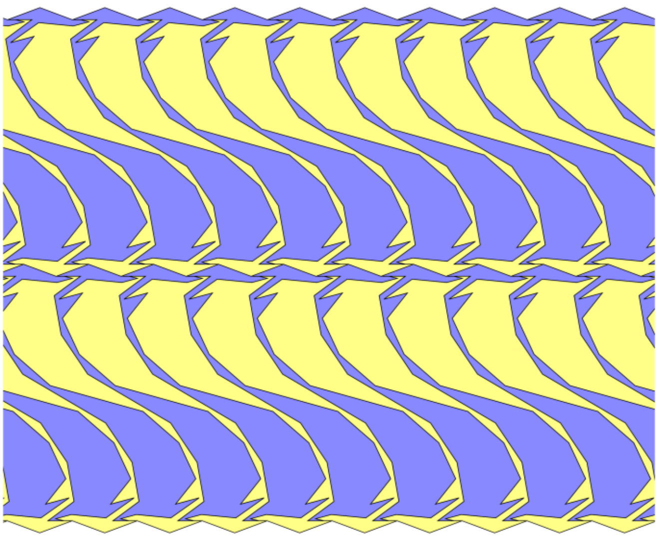

Fig. 7. Tiling the plane without flips

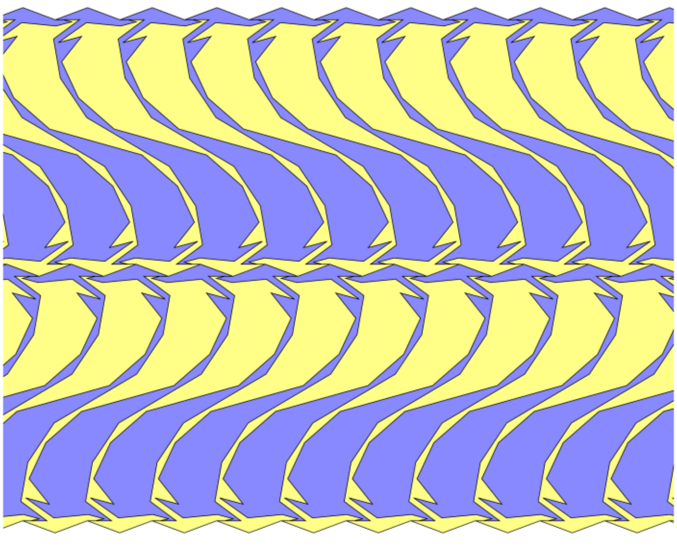

Fig. 8. Tiling the plane including flipped tiles

1.5 Construction of the new tile

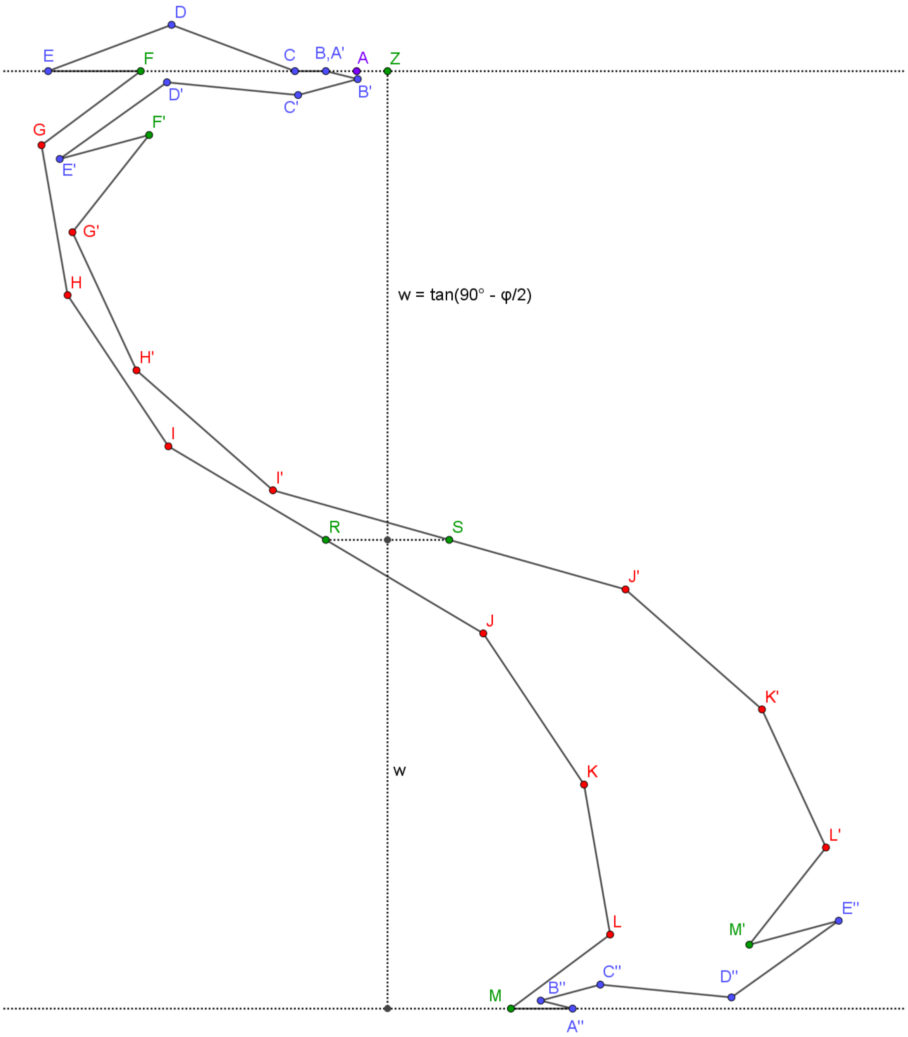

Fig. 9. Positions and names of needed points (drawn with GeoGebra [7])

The construction of the tile can be done for nearly any angle $\varphi < 25°$. If the angle is a divisor of 360°, a disk can be built with a hole in the center. Here $\varphi = 15°$ is chosen.
The triangle RSZ is isosceles with base length 2 (in any unit) and with angle φ at the apex.
The points R, S, Z, and A are not vertices of the tile.

Given are following points: Z(0, 0), R(-1, -w), S(1, -w), A(-0.5, 0), B(-1, 0), C(-1.5, 0), D(-3.5, 0.75), E(-5.5, 0), F(-4, 0), G(-5.608, -1.2), H(-5.185, -3.63), I(-3.552, -6.08)
Underlined coordinates are not fix and can be changed slightly,
but it is important that $x_D = x_C + 2$ and $x_E = x_C + 4$.

refl(P, T) is defined as the point obtained when the point P is reflected at the point T.
rot(P, T) is defined as the point obtained when the point P is rotated around T by the angle φ counterclockwise.

Determine the other points as follows:
M = refl(F, R), L = refl(G, R), K = refl(H, R), and J = refl(I, R).
B' = rot(A, Z), A' = B (!)
P' = rot(P, Z) for P = C, D, E, ..., M
P'' = refl(P', S) for P = A, B, C, D, E

In order to achieve a 25-gon leave away the points I, J, I', J'
and replace the coordinates of the point H by (-4.55, -4.777).

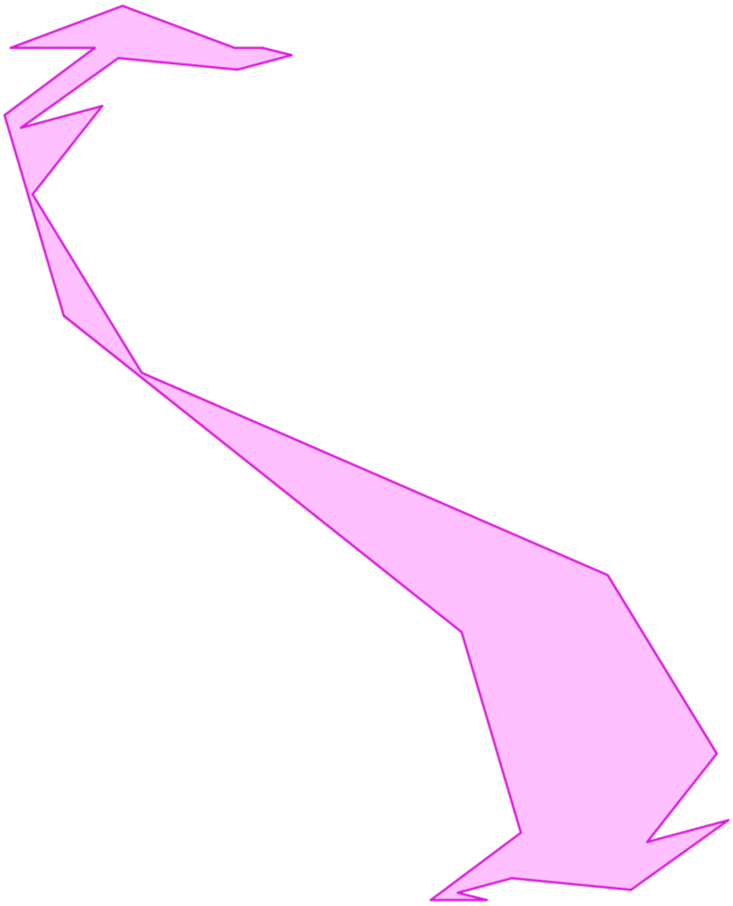

Fig. 10. A 25-gon as a thin version of the original tile

2.1 A solution for $r = 2$ that can tile the plane

This tile was found by the author already in February 2020. For more details see [5].

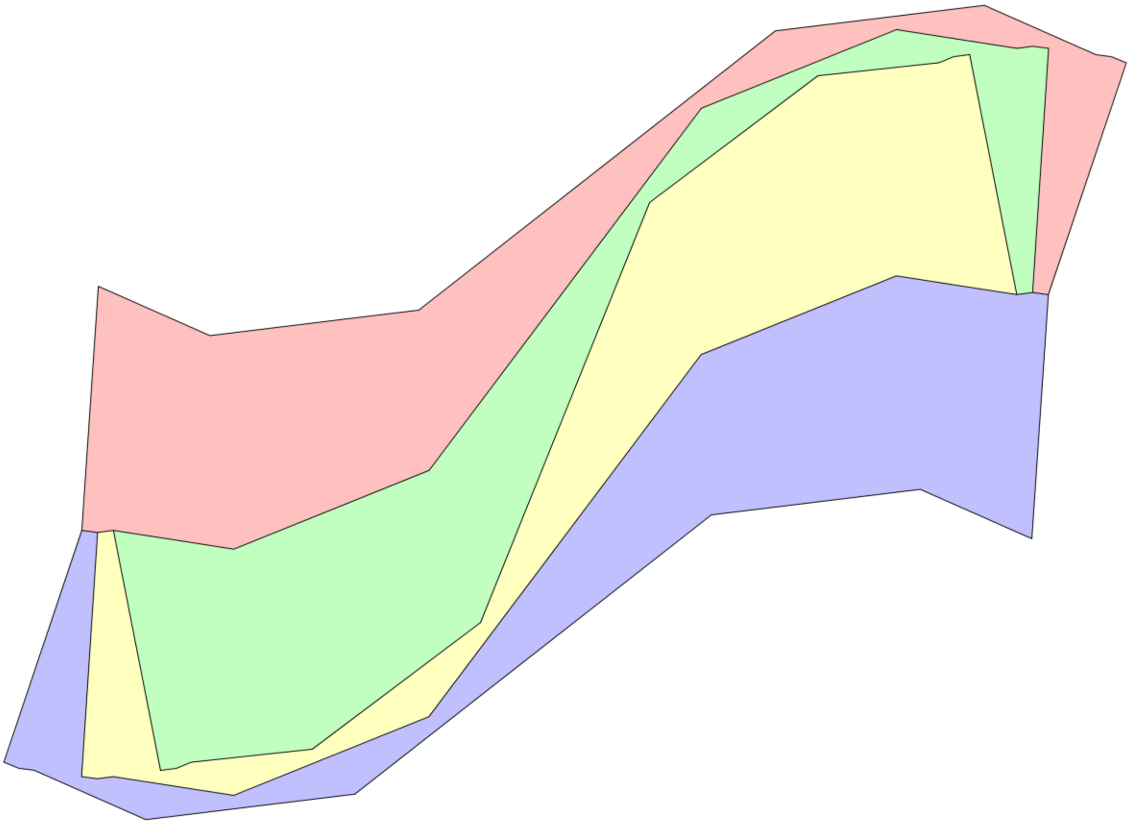

Fig. 11. A polygon as solution for $r = 2$

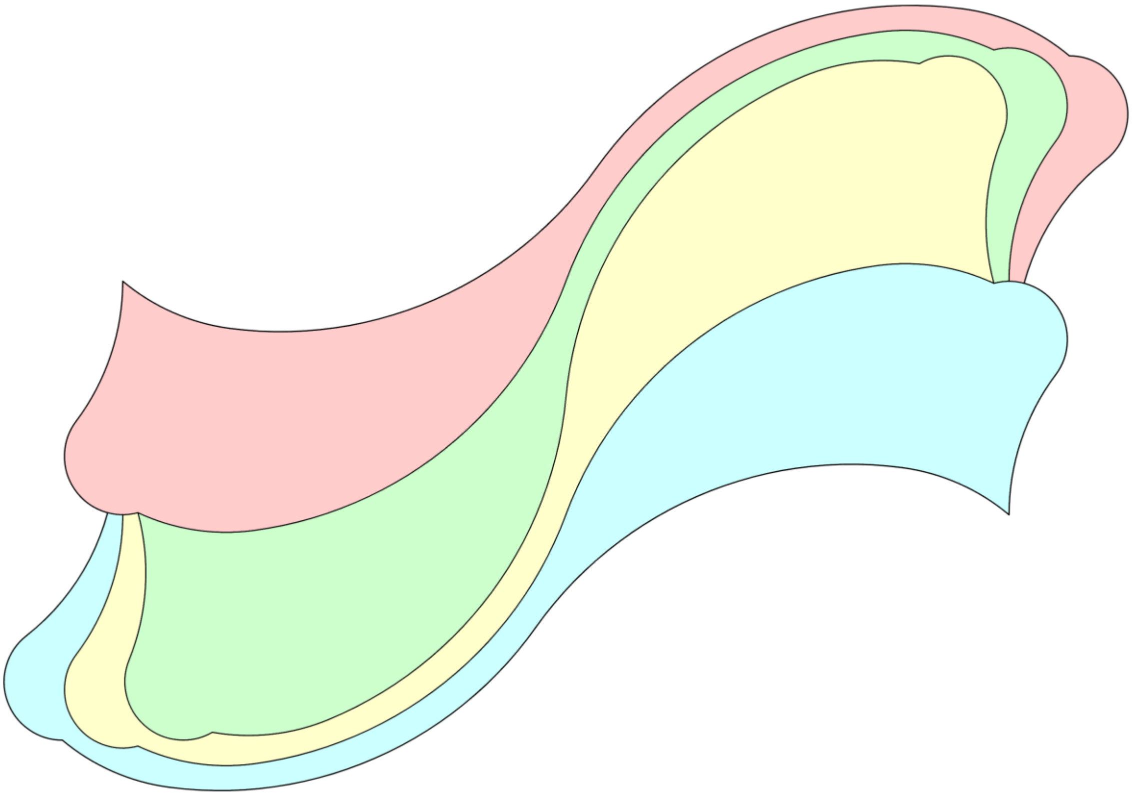

Fig. 12. A curvy solution for $r = 2$

3.1 A Solution for $r = 3$ that can be generalized all odd integers r

This shown solution was found by the author in 2024-09-08. Each tile consists of four circle arcs. Look at the bottom left part of the red tile, it shows 2 toes. The top part shows the "head" of the dragon, it consists of 2 parts with different thickness. The solution for $r = 2k - 1$ has k toes and the head consists of k parts. If you leave away the red tile, the remaing $2k$ tiles solve the enclosure problem for $r = 2k - 2$ (including the trivial case $r = 0$).

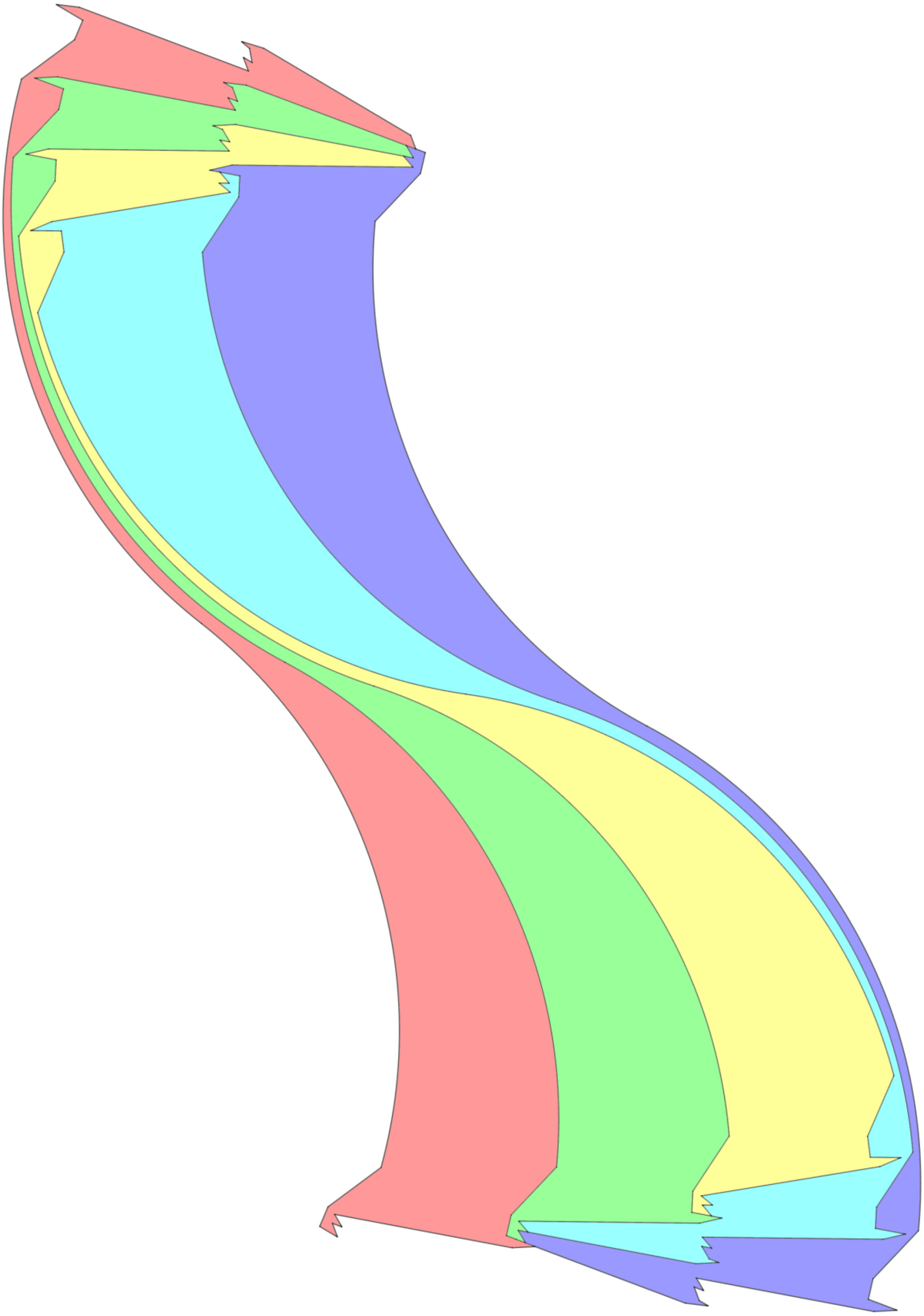

Fig. 13. A solution for r = 3

4.1 A solution for $r = 4$

The tile of this composite was found by the author in 2024-09-06. The construction is similar to the one for $r = 6$ (see chapter 5.1.). The tile does not allow a tiling of the plane.

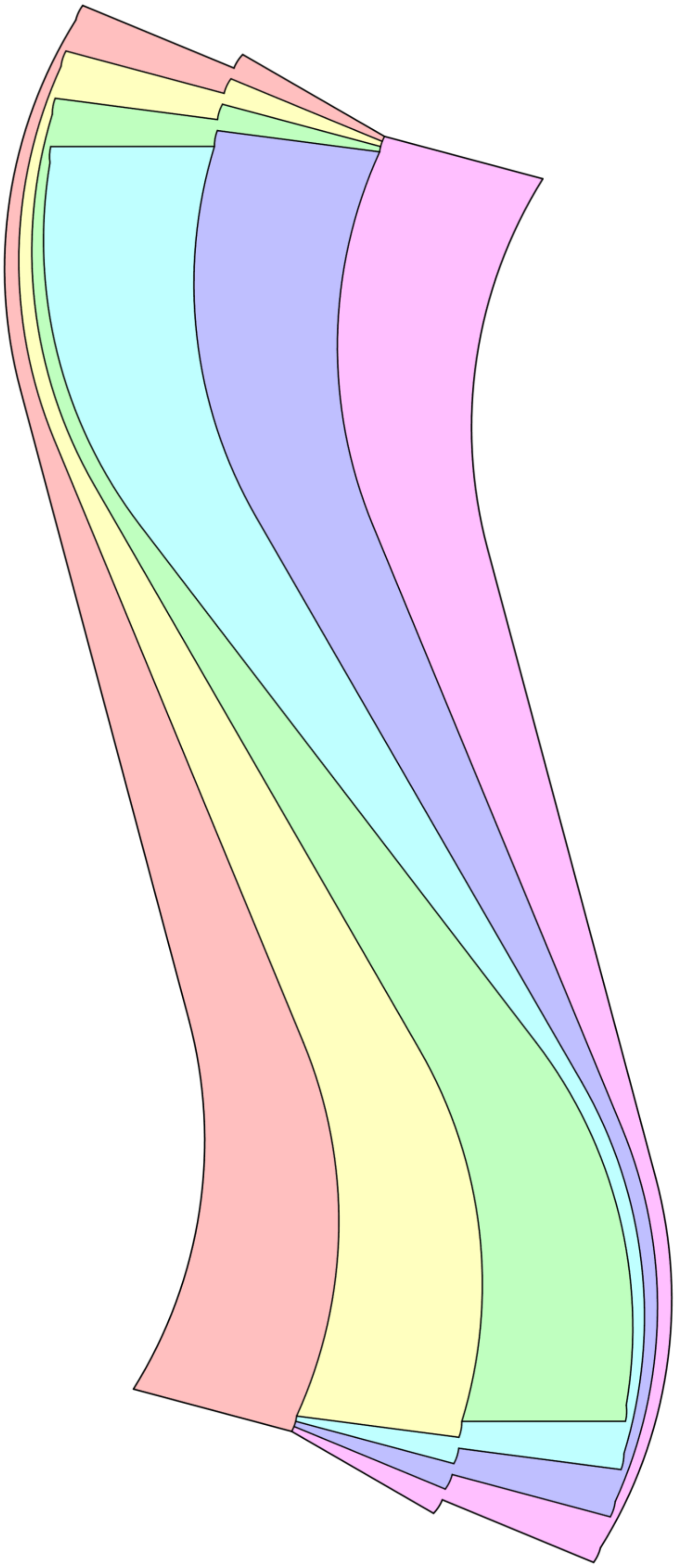

Fig. 14. A solution for $r = 4$

4.1 General solutions for $r = 2k$

The construction principle used for the shown tiling ($r = 6$) can be applied for all $r = 2k$ with positive integer k. A tiling of the plane is not possible with this tile.

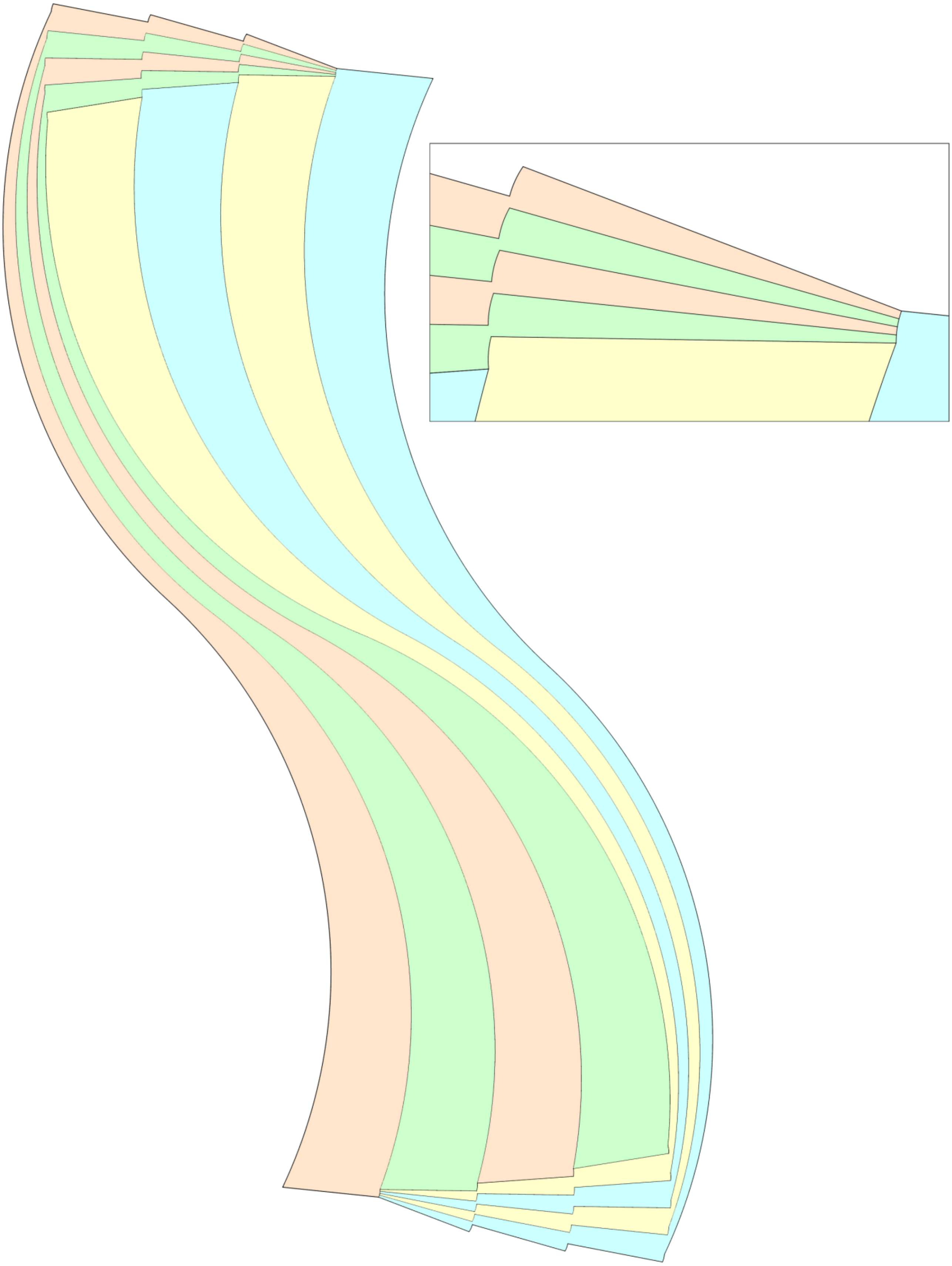

Fig. 15. A solution for $r = 6$ with an enlarged part

4.2 The construction principle for $r = 2k$

In Fig. 16, a sample with $k = 3$ is shown. The point S is the center of symmetry for the whole tiling. The dotted blue arc is part of the circle around B through A. Each tile consists of 4 large circle arcs a, $2k+1$ straight line segments s, $2k+1$ small arcs b and one tiny arc c (see Fig. 17) of length $\frac{1}{k+1}b$ with same radius as arc b.

After construction of the light green tile T_1 the other tiles can be placed like follows.
Rotate T_n around the point A by the angle φ in order to obtain the tile T_{n+1}.
In the sample $\varphi = 5°$ is used. Reflect T_n at the point S in order to obtain U_n.

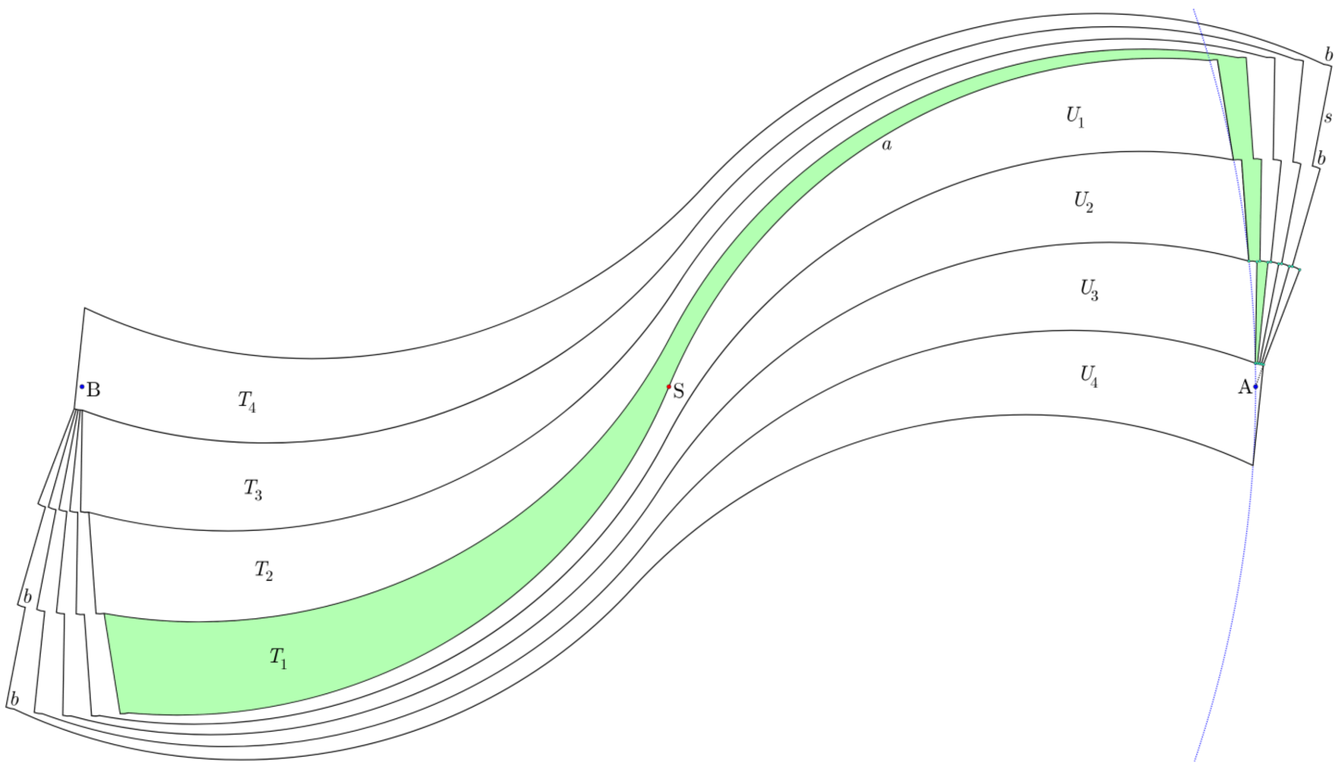

Fig. 16. Construction principle of the solution for $r = 6$

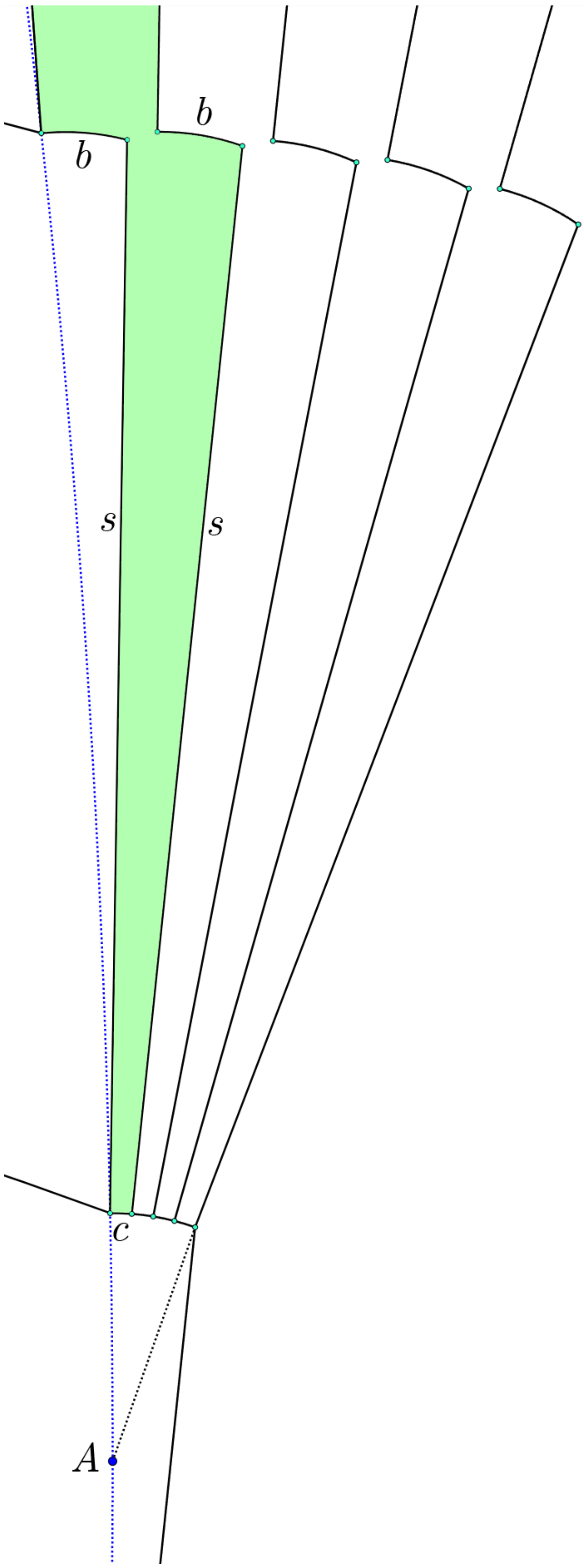

Fig. 17. Detail of the construction

Acknowledgements

Thanks to Andrejs Cibulis, Ortwin Schenker and Casey Mann for inspiring and encouraging me to search for the enclosure solutions. Thanks to Arnaud Chéritat for his marvelous tiling app [6].

References

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